Using R for Introductory Calculus and Statistics

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Background

- I have been using R for 11 years for introductory statistics.
- 5 years ago we started to revise our year-one introductory curriculum: Calculus and Statistics.
  - Calculus and Statistics topics were entirely unrelated before this.
  - Major theme of the revision was applied multivariate modeling. This ties together the calculus and statistics closely.
- We wanted a computing platform that could support both Calculus and Statistics.
- There is still resistence from faculty who do not appreciate the value of an integrated approach and who want to use a package that they are familiar with: Mathematica, Excel, SPSS, STATA

Applied Calculus: Goals

- Intended for students who do not plan to take a multi-course calculus sequence.
- Give them the math they need to work in their field of interest, rather than the foundation for future math courses they will never take.

Applied Calculus: Topics

- Change: ordinary, partial, and directional derivatives.
- Optimization: including fitting and contrained optim.
- Modeling:
  - function building blocks: linear, polynomial, exp, sin, power-law
  - functions of multiple variables
  - difference & differential equations & the phase plane
  - units and dimensions.
- Example: polynomials to 2nd order in two variables, e.g., bicycle speed as function of hill steepness and gear. There is an interaction between steepness and gear.

Introduction to Statistical Modeling: Goals

Give students the conceptual understanding and specific skills they need to address real statistical issues in their fields of interest.

- Recognize explicitly that "client" fields routinely work with multiple variables.
- ISM provides the foundations for doing so.
- Tries to provide a unified framework that applies to many different fields using different methods and terminology.
- Paradox of the conventional course:
  - It assumes that we need to teach students about t-tests, BUT ...
  - ... absurdly, that they can figure out the multivariate stuff on their own.

Introduction to Statistical Modeling: Topics

- Linear models: interpretation of terms (incl. interaction terms), meaning of coefficients, fitting
- Issues of collinearity: Simpson's paradox, degrees of freedom, etc.
- Basic inferential techniques:
  - Bootstrapping and simulation to develop concepts
  - "Black box" normal theory results
  - Anova
- Theory is presented in a geometrical framework.

Who takes these courses?

- More than 100 students each year (out of a class size of 450).
- Calculus and statistics required for the biology major.
- Economics majors take it before econometrics.
- Math majors are required to take statistics (very unusual!). They take it after linear algebra.
- About 2/3 of calculus students have had some calculus in high school.
- About 1/3 of statistics students have had an AP-type statistics course in high school.

What Makes R Effective?

- Free, multi-platform
- Powerful & integrated with graphics.
- Command-line based & modeling language
- Extensible, programmable
- Functional style, incl. lazy evaluation. This allows sensible command-line interfaces.

Example from Calculus: Functions

What students need to know about functions:

- Functions take one or more arguments and return a value.
- **Definition** of a function describes the rule.
- **Application** of a function to arguments produces the value.

R supports definition with little syntactical overhead

```
f = function(x){ x^2 + 2*x }
```

and application is very easy

```
> f(3)
[1] 15
```

R emphasizes that the function itself is a thing, distinct from its application:

```
> f
function(x){ x^2 + 2*x }
```

Functions: What's missing

Simple support for multivariate functions with vector arguments, e.g.
It would be nice to be able to say,

```
f = function([x,y,z]){ x^2 + 2*x*y + sqrt(z)*x }
```

Currently, I have to say

```
f = function(v){ v[1]^2 + 2*v[1]*v[2] + sqrt(v[3])*v[1] }
```

This isn't terrible, but it's hard to read and introduces more syntax and concepts (e.g., indexing)

Vectors: What's Missing?

- Simple, concise operations for assembling matrices. It's ugly to say:

```
> M = cbind( rbind(1,2,3), rbind(6,5,4) )
     [,1] [,2]
[1,]    1    6
[2,]    2    5
[3,]    3    4
```

- MATLAB-like consistency. If you extract a column from a matrix, it should be a column. NOT

```
> M[,1]
[1] 1 2 3
```

Example from Calculus: Differentiation

What students need to know about the derivative operator.

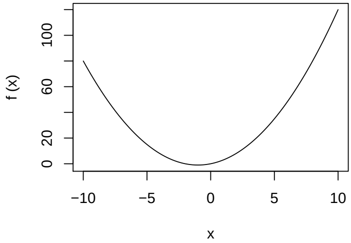
- Takes a function as input, produces a function as output.
- The output function gives the slope of the input function at any point.
- NOT PRIMARILY:
  - Algebraic algorithms for transforms: e.g., $x^n \to nx^{n-1}$
  - The theory of the infinitesimal.

A simple differentiation operator:

```
D = function(f,delta=.000001){
  function(x){ (f(x+delta) - f(x-delta))/(2*delta)} }
```

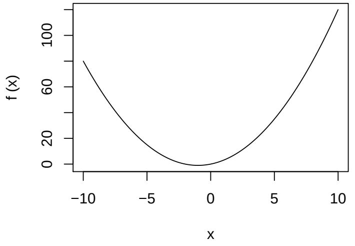
Using D

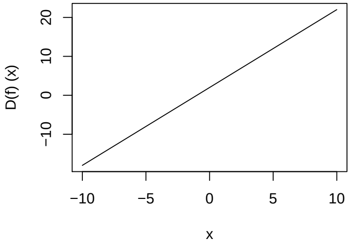
```
> f = function(x){ x^2 + 2*x }
> plot(f, 0, 10)
```



Using D

```
> f = function(x){ x^2 + 2*x }
> plot(f, 0, 10)
> plot(D(f), 0, 10)
```





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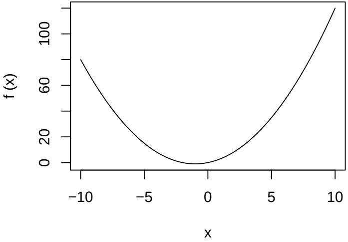
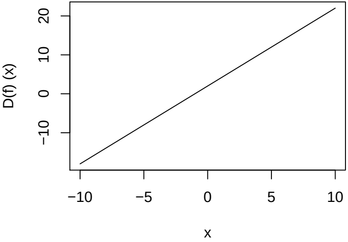
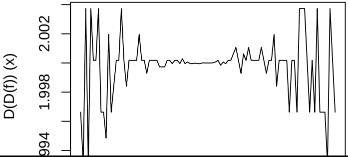
```





Numerical pathology of `(D(D(f)))`

```
> plot(D(D(f)), 0, 10)
```



Why not the built-in D?

- It doesn't reinforce the notion of an operator on functions.
- It's too complicated.

```
> g = deriv( ~ sin( 3*x), 'x')
> g
expression({
    .expr1 <- 3 * x;  .value <- sin(.expr1)
    .grad <- array(0, c(length(.value), 1), list(NULL,
    .grad[, "x"] <- cos(.expr1) * 3; attr(.value, "grad
    .value})
> x = 7
> eval(g)
[1] 0.8366556
attr(,"gradient")
             x
[1,] -1.643188
```

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            x
[1,] -1.643188
```

I need to understand better the relationship between functions and formulas, and operations on formulas for extracting structure.

Example: Fitting Linear Models

R makes this amazingly easy.

```
> g = read.csv('galton-heights.csv')
  family father mother sex height nkids
1      1   78.5   67.0   M   73.2     4
2      1   78.5   67.0   F   69.0     4
...
6      2   75.5   66.5   M   72.5     4
 and so on
> lm( height ~ sex + father, data=g)
(Intercept)         sexM       father
    34.4611       5.1760       0.4278
> lm( height ~ sex + father + mother, data=g)
(Intercept)         sexM       father       mother
    15.3448       5.2260       0.4060       0.3215
```

Operating on the results of linear modeling

Sum of squares relationship:

```
> sum( g$height^2)
[1] 4013892> m1 = lm( height ~ sex + father, data=g)
> sum( m1$fitted^2) + sum( m1$resid^2)
[1] 4013892
> m2 = lm( height ~ sex + father + mother, data=g)
> sum( m2$fitted^2) + sum( m2$resid^2)
[1] 4013892
```

Orthogonality of fitted and residual

```
> sum( m2$fitted * m2$resid )
[1] 4.239498e-12   -- essentially 0
```

Modeling: What's missing

Syntax is not forgiving of small mistakes:

- Mis-spelled column name:

```
> sum( g$heights )
[1] 0
> sum( g$height )
[1] 59951.1
```

- Named argument confounding. You flip 50 fair coins. Where's the 10th percentile on the number of heads?

```
> qbinom( .10, size=50, prob=.5)
[1] 20
> qbinom( .10, size=50, p=.5)
[1] 5
```

Standard summaries are very easy

```
> m3 = lm( height ~ sex + father + mother + nkids, data=g)
> summary(m3)
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 16.18771    2.79387   5.794 9.52e-09
sexM         5.20995    0.14422  36.125  < 2e-16
father       0.39831    0.02957  13.472  < 2e-16
mother       0.32096    0.03126  10.269  < 2e-16
nkids       -0.04382    0.02718  -1.612    0.107
> anova(m3)
             Df  Sum Sq Mean Sq    F value Pr(>F)
(Intercept)   1 4002377 4002377 8.6392e+05 <2e-16
sex           1    5875    5875 1.2680e+03 <2e-16
father        1    1001    1001 2.1609e+02 <2e-16
mother        1     490     490 1.0581e+02 <2e-16
nkids         1      12      12 2.5992e+00 0.1073
Residuals   893    4137       5
```

Note: I added the Intercept term to the ANOVA table. R lets me

Extensibility is important to teaching

Example 1: the t-test, ANOVA, and regression.
I want to show these are different aspects of the same thing.

```
> t.test(g$height)
t = 558.37, df = 897, p-value < 2.2e-16
> summary( lm( height ~ 1, data=g ) )
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  66.7607     0.1196   558.4   <2e-16
> anova( lm( height ~ 1, data=g ) )
            Df  Sum Sq Mean Sq F value    Pr(>F)
(Intercept)   1 4002377 4002377  311777 < 2.2e-16
Residuals   897   11515      13
> sqrt(311777)
[1] 558.37
```

Similarly with the 2-sample t-test

```
> t.test( g$height ~ g$sex, var.equal=TRUE)
t = -30.5481, df = 896, p-value < 2.2e-16
> summary(lm( height ~ sex, data=g))
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  64.1102     0.1206  531.70   <2e-16
sexM          5.1187     0.1676   30.55   <2e-16
> anova(lm( height ~ sex, data=g))
             Df  Sum Sq Mean Sq   F value    Pr(>F)
(Intercept)   1 4002377 4002377 635783.45 < 2.2e-16
sex           1    5875    5875    933.18 < 2.2e-16
Residuals   896    5640       6
> sqrt(933.18)
[1] 30.54800
```

Extensibility is important: Example 2

How ANOVA Works.

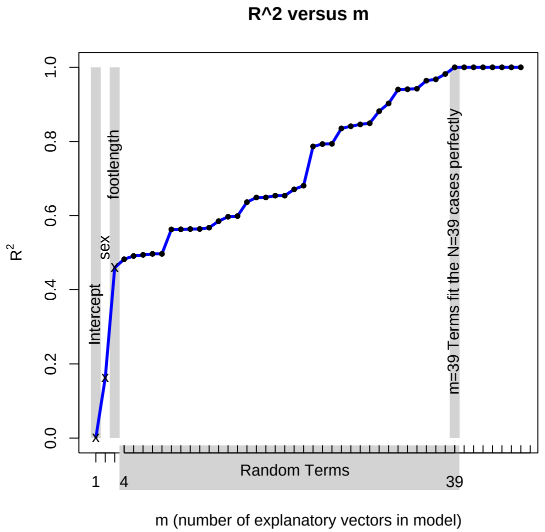
Let's add k random, junky terms to a model and see how R^2 or the fitted sum of squares changes.

`rand(k)` notation added to modeling language.

Model	R^2	ΔR^2
`footwidth~1+sex+footlength`	0.4596	
`footwidth~1+sex+footlength+rand(1)`	0.4824	0.02284
`footwidth~1+sex+footlength+rand(2)`	0.4911	0.00873
`footwidth~1+sex+footlength+rand(3)`	0.4941	0.00297
... and so on ...		
`footwidth~1+sex+footlength+rand(34)`	0.9676	0.00365
`footwidth~1+sex+footlength+rand(35)`	0.9820	0.01440
`footwidth~1+sex+footlength+rand(36)`	1.0000	0.01799
`footwidth~1+sex+footlength+rand(37)`	1.0000	0.00000
`footwidth~1+sex+footlength+rand(38)`	1.0000	0.00000

The Modeling Walk

A model with 3 model terms fit to data with 39 cases.



Resampling

Resampling itself is a conceptually simple operation.

```
> resample( c(1,2,3), 10)
 [1] 1 3 1 1 3 3 3 1 1 2

> resample( g, 5)
     family father mother sex height nkids
282      70   70.0   65.0   F   62.5     5
74       20   72.7   69.0   F   66.0     8
149      40   71.0   66.0   M   71.0     5
282.1    70   70.0   65.0   F   62.5     5
61       17   73.0   64.5   F   66.5     6
```

Repetition is conceptually simple, but ...

... generally hard for neophytes to implement on the computer.
Not in R!
Example: Roll three dice and add them.

```
> sum( resample( 1:6, 3) )
[1] 8
```

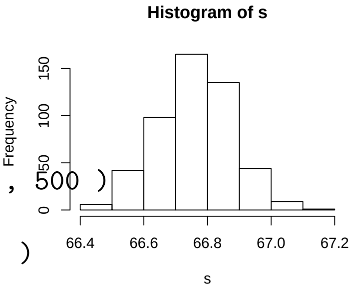
Now do this 50 times:

```
> repeattrials( sum( resample( 1:6, 3) ), 50 )
 [1] 14  6 12 10  7 13 13 11 13 10 11  6  7  5 16 14 11 13
[19] 16  7  7  9  6 10  8 10  7 15 10 14 12 14  8 11  4 10
[37] 14 10 12 10  8 12 12  8  7  4 17 16 10 11
```

Bootstrapping

Bootstrapping is hardly ever done in introductory statistics courses, even though it is so simple conceptually. This is because there is little computational support beyond the black-box type.

```
> mean( resample( g$height ) )
[1] 66.64577
> mean( resample( g$height ) )
[1] 66.76303
> s = repeattrials(
  mean( resample( g$height ) ), 500 )
> hist(s)
> quantile( s, c(0.025, 0.975) )
    2.5%    97.5%
66.52771 66.97620
```



A command-line interface has big advantages

It allows us to put things together in creative ways.

Example 1: Confidence intervals on model coefficients.

```
> lm( height ~ sex + nkids, data=g )
(Intercept)         sexM       nkids
    64.8013       5.0815     -0.1095
> lm( height ~ sex + nkids, data=resample(g) )
(Intercept)         sexM       nkids
   64.73765      5.15831    -0.09852
> s = repeattrials(lm( height ~ sex + nkids,
          data=resample(g) )$coef, 1000)
> head(s)
  (Intercept)     sexM      nkids
1    65.01683 5.323394 -0.1664674
2    64.64250 5.262300 -0.1005491
3    64.75436 5.113593 -0.1079453
  and so on
> quantile( s$nkids, c(0.025, 0.975))
```

Resampling: Example 2

Hypothesis testing on single variables:

```
> lm( height ~ sex + nkids, data=g )
(Intercept)          sexM        nkids
    64.8013        5.0815      -0.1095
> lm( height ~ sex + resample(nkids), data=g )
    (Intercept)                sexM  resample(nkids)
       64.00688             5.12503          0.01628
> s = repeattrials(lm( height ~ sex + resample(nkids),
    data=g )$coef, 1000)
> head(s)
  (Intercept)     sexM resample(nkids)
1    63.99812 5.117672      0.01821168
2    64.18064 5.119589     -0.01154208
   and so on
> quantile( s[,3], c(0.025, 0.975))
       2.5%       97.5%
-0.05690810  0.05361429
```

Resampling: Example 3

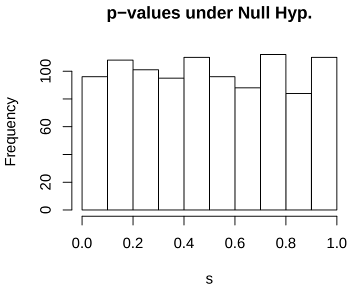
Power/Sample-size demonstration. If the world were like our sample, how likely is a sample of 100 people to demonstrate that family size (`nkids`) is related to height?

```
# Extract the p-value on nkids
> anova( lm(height ~ sex + nkids, data=g))[3,5]
[1] 0.0004454307
# Simulate a sample of size 100
> anova( lm(height ~ sex + nkids, data=resample(g,100)))[3,
[1] 0.2743715
> s = repeattrials(anova( lm(height ~ sex + nkids, data=res
> head(s)
[1] 0.001870581 0.498089249 0.801042654 0.286201801
[5] 0.055200572 0.198855304 and so on
> table( s < .05 )
FALSE  TRUE
  774   226   # power is 23%
```

Distribution of p-values

Under the null:

```
> s = repeattrials(
  anova( lm(height ~ sex + resample(nkids), data=g))[3,5],
  1000)
```



It would be nice to have a GUI that can support this kind of thing. How?

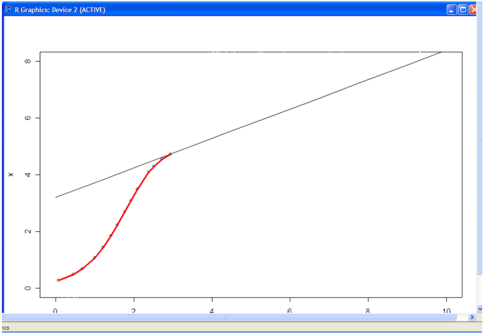
GUIs are Important

Examples from our courses:

- Euler method of integration.
- Visualizing dynamics on the phase plane.
- Linear combinations of vectors.

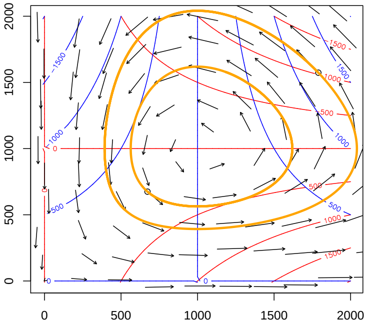
future simulating causal networks.

A graphical approach to integration



The logistic-growth system:
$\dot{x} = rx(1 - x/K)$

- The differential equation describes **local** dynamics.
- Growth rate changes with x.
- Accumulate small increments.



It's also calculus to teach the phenomenology of differential equations:

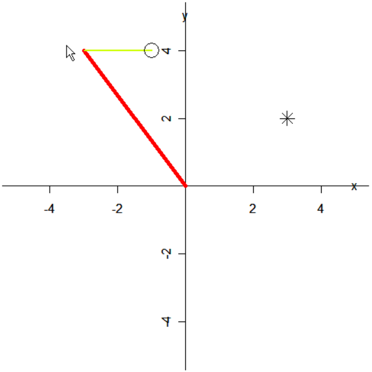
- equilibrium and stability
- oscillation

Computers can solve the DEs, so solution techniques are no longer central.

Fitting Linear Models

A	B	C
3	-3	2
2	4	0

Fit the model `A ~ B + C - 1`



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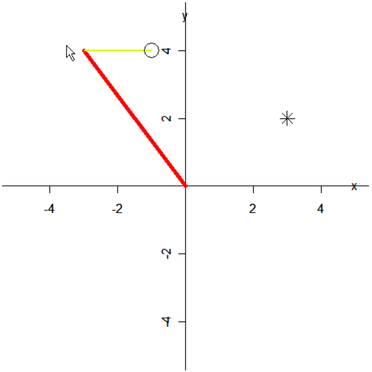

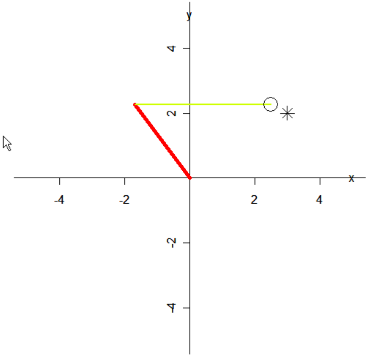

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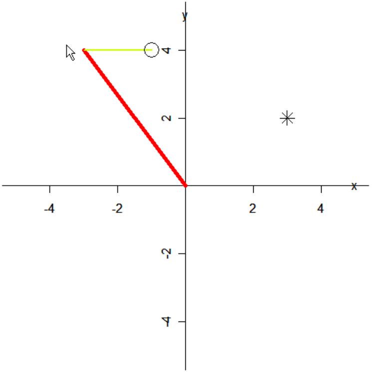

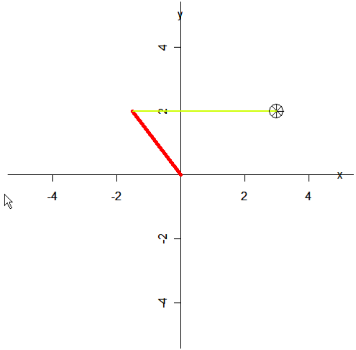

Local Requirements for Adopting R

- A locally accessible expert.
- Concise instructions on how to do basic things. Like Kermit Sigmon's MATLAB Primer.
- Things are vastly better than they once were, but still we don't exploit the 80/20 rule:
  20% of the knowledge will get you 80% of the way there!

Summary

- GUIs are important, but ...
- We should embrace R's strength, an extensible command-line interface and syntax.

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